

# Universal Character of Stochastic Resonance and a Constructive Role of White Noise

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It is shown that, in spite of claims put forward in the literature, the stochastic resonance (SR) appears even in linear systems—both overdamped and inertial—driven by Gaussian white noise, and even after averaging the asymptotics over the initial phase of the input signal. This supports recent suggestions that SR is a universal effect present in every stochastic process modulated by external signals. It is also shown that the noise may sustain the output signal which otherwise would vanish exponentially in the course of time.

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**KEY WORDS:** Stochastic resonance; linear systems; Gaussian white noise.

## 1. INTRODUCTION

The idea of *constructive role of noise and fluctuations*, although seemingly paradoxical, has gained recently considerable attention.<sup>(1)</sup> The best known examples of phenomena in which random forces play a constructive role are the stochastic resonance (SR)<sup>(2,3)</sup> and, related to it,<sup>(4)</sup> molecular motors (Brownian ratchets).<sup>(5)</sup>

SR is a phenomenon in which the response of a dynamical system to an input signal is optimized by the presence of a specific level of noise. The standard criterion for SR is the appearance of a peak in the signal-to-noise ratio (SNR) (in the power spectrum) vs. input noise strength, whereas the physics of this phenomenon is a significant change of some characteristics, like the transmission of matter or information, due to the transfer of energy from the stochastic field into some physical process, stimulated by external

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pumping noise, and appearing in some more or less narrow range of noise intensity.

It has been believed for a long time that there are three essential ingredients necessary for the onset of SR: bistability of the dynamical system, periodic input, and intrinsic random forces. However, discovery of aperiodic SR<sup>(3,6)</sup> proved that also non-periodic signals can be enhanced by the SR mechanism, and therefore the *periodic* forcing is not an essential ingredient. SR has been also detected in monostable nonlinear systems,<sup>(7)</sup> and in excitable systems with<sup>(8)</sup> and without<sup>(9,10)</sup> threshold crossing; therefore bistability appears not to be essential, either. It has been discovered recently that, unexpectedly, even nonlinearity is not necessary—the presence of SR in *linear* enzymatic kinetics nonlinearly coupled both to a periodic field and to noise was first suggested in ref. 11 and later discussed in more detail in ref. 12. Biophysical and biochemical enzymatic processes are highly complex and, in general, nonlinear. However, there is a commonly used representation, the so-called Michaelis–Menten scheme, which is linear in the enzyme probability states, nonlinearities being hidden in time-dependent coefficients. This scheme can be rewritten as the linear kinetics with both multiplicative and additive noise driving. First, explicit demonstration of SR in a simple linear system with multiplicative (parametric) noise was given in ref. 13 and discussed more thoroughly in refs. 14 and 15. Besides, linear SR (both periodic and aperiodic) has been proposed as a plausible explanation<sup>(16)</sup> of experimental data of active transport of  $\text{Na}^+$  and  $\text{Rb}^+$  in human erythrocytes catalyzed by  $\text{Na}^+ - \text{K}^+ - \text{ATPase}$ ,<sup>(17)</sup> although the SR mechanism in the latter case is more complicated than in the standard one: the noise intensity depends here on the intensity of the external driving.<sup>(16)</sup> All these recent developments demonstrate that the SR phenomenon does not depend on the barrier or threshold crossing, and even on a nonlinear character of the process. Indeed, the SR has been recently claimed to be just an inherent property of rate-modulated random series of events.<sup>(10)</sup> The discovery of the linear SR (LSR) suggests that SR is quite a universal phenomenon—the linear (relaxation) kinetics is the final stage of most dissipative processes, and therefore the presence of SR in the linear stage suggests the appearance of SR in earlier, nonlinear stages as well. However, current literature claims that the appearance of LSR is restricted: the noise needs to be multiplicative, the effect vanishes in the Gaussian white noise (GWN) limit,<sup>(14)</sup> and vanishes after averaging over the initial phase of the input signal (driving field).<sup>(15)</sup> The aim of this report is to demonstrate that SR is a still more universal phenomenon than can be judged from the up-to-date literature. In other words, for almost any process driven by an external signal and internal or external noise one can find some characteristics of the process, like the amplitude of the outgoing

signal, or current, or the higher moments, or correlation functions, or some combinations of such quantities, if not the direct SNR response itself, which bear the signature of SR. For this aim we shall consider linear processes driven by GWN only—it is obvious that the detection of SR due to GWN implies the appearance of SR due to colored noises, too. The calculations we present are provided to demonstrate the universality of SR, and not for the purpose of presenting yet other examples of SR.

We follow the general procedure: (i) write the formal solution  $X(t)$  to a model stochastic equation. (ii) construct the product  $X(t) X(t + \tau)$ , (iii) calculate the appropriate averages. In doing so, we repeatedly use the fact that for GWN with  $\langle \eta(t) \eta(s) \rangle = D_0^2 \delta(t - s)$

$$\langle e^{a \int_{t_1}^{t_2} \eta(t') dt'} \rangle = e^{1/2 a^2 D_0^2 (t_2 - t_1)} \tag{1}$$

and the average of a product of two functions defined on disjoint time intervals factorizes. Thus, for instance,

$$\langle e^{a \int_0^t \eta(t') dt'} e^{a \int_0^{t+\tau} \eta(t') dt'} \rangle = \langle e^{2a \int_0^t \eta(t') dt'} e^{a \int_t^{t+\tau} \eta(t') dt'} \rangle = e^{2a^2 D_0^2 t} e^{1/2 a^2 D_0^2 \tau} \tag{2}$$

This procedure produces rather lengthy intermediate formulas. For brevity, we will mostly leave out details of the calculations. Averages (2) can also be calculated for dichotomic noises, by obvious generalizations of methods used in ref. 18.

## 2. THE OVERDAMPED TRANSMITTER

Let us begin with a simple case: the driven noisy relaxation with multiplicative noise:

$$\dot{X}(t) = -(a_0 + \eta(t)) X(t) + A_i \cos(\Omega t + \phi_i) \tag{3}$$

The reason for considering such systems is that in most enzymatic processes the overdamped approximation works very well: the inertial term is estimated to be about 10 orders of magnitude smaller than the “friction” term. The flow (3) has been discussed in refs. 13–15 and 19 for  $\eta(t)$  being Gaussian and dichotomous noises. Contrary to claims put forward earlier, we shall show that signatures of SR can be found in this system both for Gaussian white noise (GWN) and after averaging over the initial phase  $\phi_i$ . As the general formulas for the averages are lengthy, consider two special cases: asymptotic state not averaged over the initial phase, and transient behavior averaged over the initial state.

The solution of Eq. (3) can be written in the form

$$\langle X(t) \rangle = f(t) + A_o \cos(\Omega t + \phi_i + \phi_o) \tag{4}$$

where  $f(t)$  describes the transient behavior, and  $A_i$ ,  $\phi_i$ ,  $A_o$ ,  $\phi_o$  are amplitudes and phases of the input and output, signals, respectively. Exact, shapes of  $f(t)$ ,  $A_o$ ,  $\phi_o$  depend on the type of noise  $\eta(t)$ . Output parameters for the GWN are<sup>(14, 15)</sup> ( $\beta = a_0 - \frac{1}{2}D_0^2$ )

$$A_o = A_i / \sqrt{\beta^2 + \Omega^2}, \quad \tan \phi_o = -\Omega / \beta \quad (5)$$

and the system is convergent if and only if  $\beta \geq 0$ . According to refs. 14 and 15, this system exhibits SR for colored noises only. However, it is easy to verify that  $A_o$  has a maximum at  $\frac{1}{2}D_0^2 = a_0$ , i.e., at the limit of the convergence of the process  $\langle X(t) \rangle$ , and, moreover, that the amplitude-SNR, defined<sup>(14, 15)</sup> as  $R_A = A_o / A_i D_0^2$  has two extrema at  $\frac{1}{2}D_0^2 = \frac{1}{4}(3a_0 \pm \sqrt{a_0^2 - 8\Omega^2})$ , both corresponding to the convergent  $\langle X(t) \rangle$ . This effect, however, indeed vanishes after averaging over the initial phase, because of vanishing of the oscillating component in Eq. (4).

Most direct definition of SNR is via the power spectral density  $S(\omega)$  (cf. [ref. 1, Eqs. (2.12)–(2.13)]). In general  $S(\omega)$  can be written in the form:

$$S(\omega) = \int_0^\infty dt \cos(\omega\tau) \langle X(t) X(t+\tau) \rangle \quad (6)$$

which for the system (3) gives

$$S(\omega) = \Psi_0(t) \frac{\beta}{\beta^2 + \Omega^2} + \frac{1}{2} \pi \Psi_c(t) [\delta(\omega + \Omega) + \delta(\omega - \Omega)] + \Psi_s(t) \frac{\omega}{\beta^2 - \Omega^2} \quad (7)$$

with the standard definition of SNR as

$$R_S = \Psi_c \left/ \left( \Psi_0 \frac{\beta}{\beta^2 + \Omega^2} \right) \right|_{\omega = \pm \Omega} \quad (8)$$

Note, that since the flow  $X(t)$  is nonstationary, the spectrum (7) does depend on both  $t$  and  $\tau$ . It is, nevertheless, a direct observable quantity, and procedures for measuring power spectra of nonstationary signals are well known.<sup>(20)</sup>

General formulas for  $\Psi_0$ ,  $\Psi_c$  are lengthy, but asymptotically (i.e., omitting terms which vanish in the limit  $t \rightarrow \infty$ ) we get

$$\Psi_0^\infty = \frac{(\beta - \alpha)(\sqrt{(\alpha + \beta)^2 + 4\Omega^2} - (\alpha + \beta) \cos(2\Omega t + 2\phi_i + 2\tilde{\phi}))}{2(\alpha + \beta)(\beta^2 + \Omega^2) \sqrt{(\alpha + \beta)^2 + 4\Omega^2}} \quad (9)$$

$$\Psi_c^\infty = \frac{1 + \cos(2\Omega t + 2\phi_i + 2\tilde{\phi})}{2(\beta^2 + \Omega^2)} \quad (10)$$

where

$$\tan(2\tilde{\phi}) = -\frac{2\Omega(\Omega^2 - \beta(\alpha + 2\beta))}{(\alpha + 5\beta)\Omega^2 - (\alpha + \beta)\beta^2} \quad (11)$$

$$\tan(2\tilde{\varphi}) = -\frac{2\beta\Omega}{\beta^2 - \Omega^2} \quad (12)$$

$$\alpha = a_0 - \frac{3}{2}D_0^2 \quad (13)$$

Note that the power spectrum (7), or indeed the variance of the process (4), is properly defined only if  $a_0 - D_0^2 = \frac{1}{2}(\alpha + \beta) \geq 0$ .

Asymptotic SNR  $R_s^\infty$  exhibits SR for some (rather narrow) time range — see Fig. 1. The effects repeats itself with the period  $\pi/\Omega$  and vanishes after averaging over the initial phase. On the other hand, the averaging over the initial phase does not destroy the non-asymptotic (transient) SR; we show this transient effect in Fig. 2.

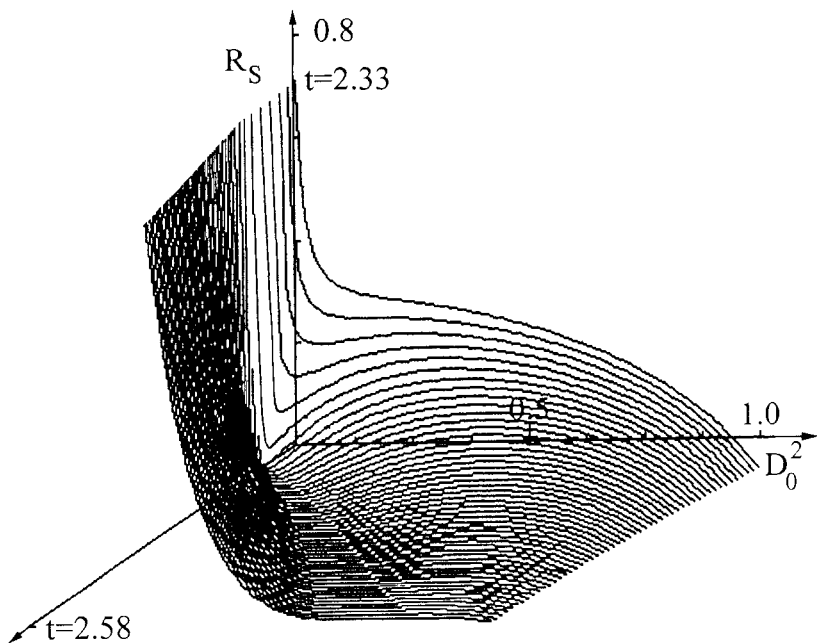


Fig. 1. Asymptotic ( $t \rightarrow \infty$ ) signal-to-noise ratio  $R_S$ , Eq. (8). The effect repeats itself in time with period  $\pi/\Omega$ .

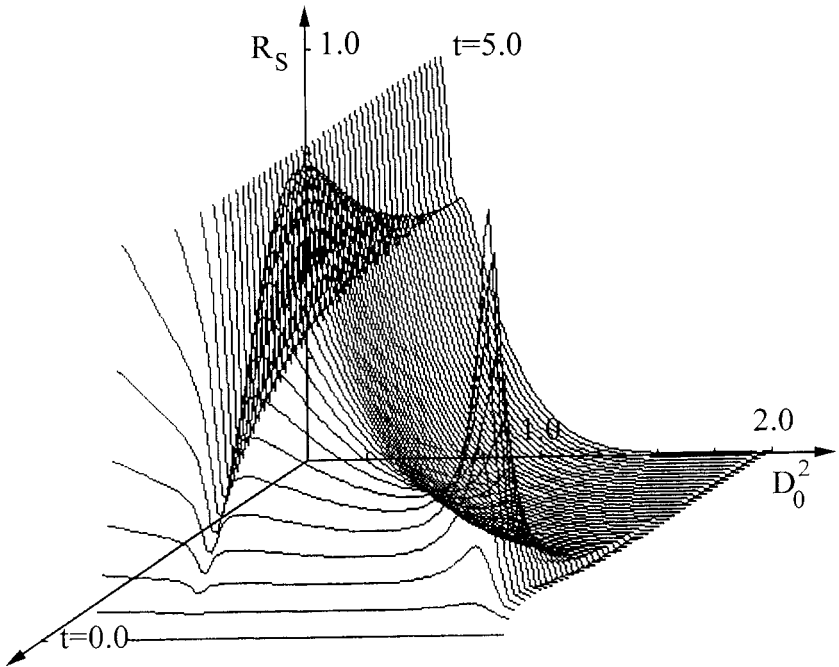


Fig. 2. Transient  $R_S$ , Eq. (8), for linear process (3), averaged over the initial phase.  $a_0 = 1$ ,  $\Omega = 0.05$ . The effect vanishes for higher frequencies  $\Omega$ .

On the other hand, when the input signal is coupled parametrically to the process  $X(t)$  and the noise is additive:

$$\dot{X}(t) = -[a_0 + A_i \cos(\Omega t + \phi_i)] X(t) + \eta(t) \quad (14)$$

the power spectrum has the form  $S(\omega; t) = S_D + D_0^2 S_N$ , with  $S_N \rightarrow 0$  for  $t \rightarrow \infty$  which implies the lack of conventional SR. (A nonlinear analog of Eq. (14) leads to the monostable SR.<sup>(7)</sup>) Nevertheless there is an interesting, never described so far, constructive effect of the presence of noise:  $S_N$  survives both the limit  $t \rightarrow \infty$  and the  $\phi_i$ -averaging, i.e., the presence of noise sustains the output signal (at frequency  $\Omega$  and at higher harmonics!), which otherwise, for a purely deterministic case, would vanish exponentially in the course of time. Although there is no resonance—here is no optimal noise level—this is still undoubtedly a cooperative effect between the noise, the driving field, and the intrinsic (relaxation) kinetics. This aspect will be discussed in detail elsewhere.

The above effect seems to explain the appearance and persistence of SR in the “fully mixed” linear case:<sup>(11, 12, 16)</sup>

$$\dot{X}(t) = [f_1(t) + f_2(t) \eta(t)] X(t) + f_3(t) + f_4(t) \eta(t) \tag{15}$$

where  $f_i(t)$  are periodic functions of time, generated by a common external periodic field. Such fully mixed model described, among others, the linear enzymatic kinetics, and reproduces correctly<sup>(16)</sup>—in terms of SR between the external field and the fluctuations of the membrane potential—the experimental data on active transport of ions (catalyzed by  $\text{Na}^+ - \text{K}^+ - \text{ATPase}$ ) through biological membrane of human erythrocytes.<sup>(17)</sup> In the model (15) the SR appears both for the averaged and for instantaneous current in the asymptotic state, and both for noise level (fluctuations of the membrane electric potential) induced in part by the external field<sup>(16)</sup> and for noise level independent of the external field.<sup>(12)</sup> These results prove, that the standard linear SR exists also in the averaged asymptotic state.

### 3. THE INERTIAL PROCESS

Consider now another example, the inertial process:

$$\begin{cases} \dot{X} = (1 + \eta(t)) V \\ \dot{V} = -\Omega^2(1 + \eta(t)) X + b_0 \end{cases} \tag{16}$$

In the absence of noise, this is equivalent to the equations of motion of a harmonic oscillator. There is no periodic driving in this case, and we may only think of autonomous<sup>(21)</sup> SR. The formal solution for  $X(t)$  reads:

$$X(t) = X_0 \cos[\Omega\gamma(t, 0)] + \frac{V_0}{\Omega} \sin[\Omega\gamma(t, 0)] + \frac{b_0}{\Omega} \int_0^t \sin[\Omega\gamma(t, t')] dt' \tag{17}$$

where

$$\gamma(t, t') \equiv t - t' + \int_{t'}^t \eta(z) dz \tag{18}$$

While calculating the appropriate averages for the process (17), we repeatedly encounter expressions similar to

$$\begin{aligned} \langle \sin[\Omega\gamma(t, t')] \rangle &= \frac{1}{2i} \{ \langle e^{i\Omega(t-t')} e^{i\Omega \int_{t'}^t \eta(z) dz} \rangle - \langle e^{-i\Omega(t-t')} e^{-i\Omega \int_{t'}^t \eta(z) dz} \rangle \} \\ &= e^{-1/2 \Omega^2 D_0^2(t-t')} \sin \Omega(t-t') \end{aligned} \tag{19}$$

$$\langle \cos[\Omega\gamma(t, t')] \rangle = e^{-1/2 \Omega^2 D_0^2(t-t')} \cos \Omega(t-t') \tag{20}$$

(cf. Eq. (2)). The calculations are elementary though tedious, and the detailed formulas are lengthy. We will only present solutions in the asymptotic regime, i.e., excluding terms which fall off exponentially with  $t \rightarrow \infty$ . In this regime

$$\langle X(t) \rangle = \frac{4b_0}{\Omega^2(4 + D_0^4\Omega^2)} \quad (21)$$

$$\begin{aligned} \langle X(t) X(t + \tau) \rangle &= \frac{16b_0^2}{\Omega^4(4 + D_0^4\Omega^2)^2} + 2t \frac{b_0^2 D_0^2}{\Omega^2(4 + D_0^4\Omega^2)} \cos \Omega\tau e^{-1/2 D_0^2\Omega^2\tau} \\ &+ \frac{1}{2} \left( X_0^2 + \frac{V_0^2}{\Omega^2} \right) \cos \Omega\tau e^{-1/2 D_0^2\Omega^2\tau} \\ &+ \frac{2b_0}{\Omega^2(4 + D_0^4\Omega^2)} (V_0 D_0^2 - 2X_0) \cos \Omega\tau e^{-1/2 D_0^2\Omega^2\tau} \\ &+ \frac{b_0^2}{\Omega^4(4 + D_0^4\Omega^2)^2 (1 + D_0^4\Omega^2)} [(D_0^8\Omega^4 - 18D_0^4\Omega^2 - 28) \cos \Omega\tau \\ &- D_0^2\Omega(20 + 11D_0^4\Omega^2) \sin \Omega\tau] e^{-1/2 D_0^2\Omega^2\tau} \end{aligned} \quad (22)$$

We can see that while  $\langle X(t) \rangle$  goes to a constant value, the correlation function (22) diverges linearly with  $t \rightarrow \infty$ . If we define:

$$X_s(t) = X(t) - \frac{4b_0}{\Omega^2(4 + D_0^4\Omega^2)} \quad (23)$$

we can immediately see that for  $t \rightarrow \infty$  but finite

$$\langle X_s(t) \rangle = 0 \quad (24a)$$

$$\langle X_s^2(t) \rangle = \frac{2b_0^2 D_0^2}{\Omega^2(4 + D_0^4\Omega^2)} t = 2Dt \quad (24b)$$

In other words, the process  $X_s(t)$  behaves asymptotically like a simple diffusion. It is straightforward to see that from the point of view of the diffusion process (24) there is an optimal level of noise,  $D_0^2 = 2/\Omega$ , which maximizes the diffusion constant  $D$ . A similar effect can be seen in the power spectrum of (22), which after removing the  $\delta(\omega = 0)$  term corresponding to the constant (zero frequency) external driving is for  $t$  large enough dominated by

$$S(\omega) = \frac{2b_0^2 D_0^4 t}{\Omega^2(4 + D_0^4\Omega^2)} \left\{ \frac{1}{4(1 - (\omega/\Omega))^2 + D_0^4\Omega^2} + \frac{1}{4(1 + (\omega/\Omega))^2 + D_0^4\Omega^2} \right\} \quad (25)$$



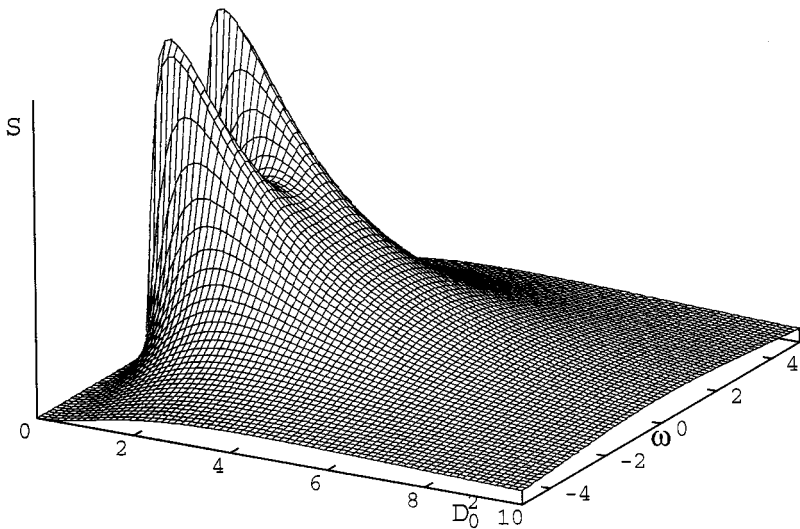


Fig. 3. The power spectrum (25) of the dominating term in the correlation (22). Time  $t$  is such that the exponentially vanishing contributions to the correlation function are negligibly small. The scale of the  $S$  axis is determined by the current value of  $t$ . Other parameters are  $b_0 = 1.0$ ,  $\Omega = 1.0$ .

(cf. Fig. 3). Again we see that the diffusion constant in (24b) and the power spectrum in the asymptotic regime both behave in a SR-like manner: they are optimized by certain well-defined levels of noise.

#### 4. DISCUSSION

We have shown that SR-like characteristics can be found even in systems so far believed to be immune to that effect. It is worth to stress that the effects reported here are induced by the Gaussian white noise, which represents standard equilibrium thermal fluctuations.

One may ask what is the “genuine” SR. Barzykin *et al.*<sup>(15)</sup> claim that it requires a maximum in the SNR ( $R_S$  in our notation) in the asymptotical state and after averaging over  $\phi_i$ . In the present authors’ opinion such requirements are much too strong: (i) Many effects vanish after averaging (e.g., after stirring in spatially extended systems), but there is no reason to deem them nonexistent or spurious. (ii) We have shown that transient SR remains even after the  $\phi_i$ -averaging. One may ask in turn whether such short-lived transient effects are relevant at all. Let us mention, therefore, that there are situations when the process  $X(t)$  is repeatedly revived, e.g., in biologically important enzymatic processes. In such cases SR will be

visible as repeating short time bursts of enhanced output signal, and also in the increase of the time-averaged output signal when the revivals are sufficiently frequent. (iii) Constructive role of noise may manifest itself not only in the signal transmission, expressed through SNR—there are other directly observable quantities like the amplitude  $A_o$ , Eq. (5), the currents in refs. 16 and 17, or the diffusion constant (24b).

One more point deserves a brief discussion: Our model systems are linear in the sense that their equations of motion are linear with time-dependent coefficients, and thus one may be tempted to call the phenomena discussed above a “linear stochastic resonance.” There is, however, a point in observation that since noise is meant to represent many unobserved and unaccounted for degrees of freedom—instead of considering impossibly complicated microscopic motions, we mimic their effect by a reasonably simple stochastic process—a multiplicative (nonlinear) coupling between the stochastic process and the observed degrees of freedom means a “hidden” nonlinearity. This is certainly the case for instance in the Michaelis–Menten scheme mentioned above. This is, in a sense, similar to a situation known from the dynamical systems theory when one system is coupled to another one which “drives” the former to a certain location in the phase space, but there is no coupling back between the “response” system and the driving one; see, for example, ref. 22 for a review.

Moreover, the responses of most the model systems discussed here to the multiplicative stochastic term and to the external deterministic driving are not separable—the time behavior resulting from the two signals (deterministic and stochastic) acting together is not a linear combination of behaviors resulting from each of them acting separately. It would be proper to call a system “linear” if its response to various forces is decomposable.

The conclusion of this report may be formulated as follows: SR effect is present everywhere the stochastic process is modulated by external fields (“signal”), although it may manifest itself through different quantities. This observation supports and strengthens recent Bezrukov’s claim<sup>(10)</sup> that SR is “an inherent property of rate-modulated series of events.”

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